Spherical Wavelets

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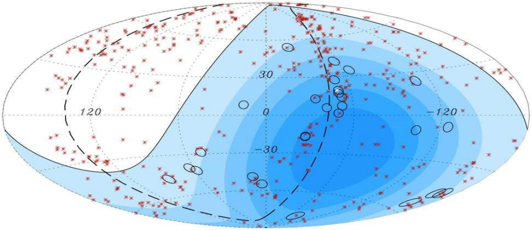
Motivation

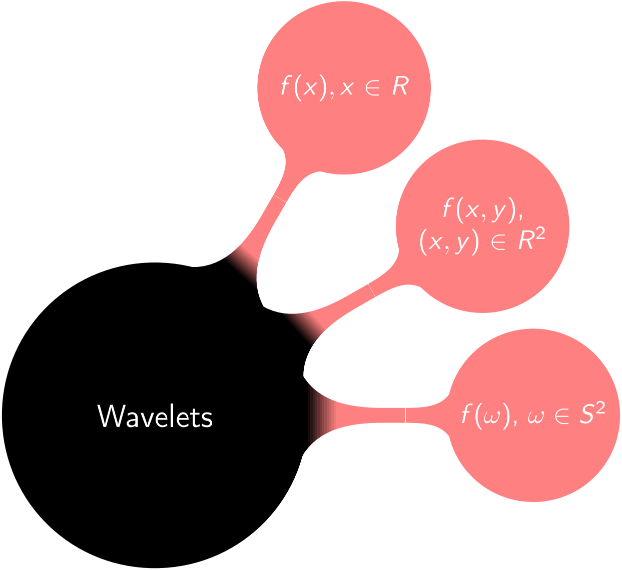
When are wavelets useful?

"... if we are interested in transient phenomena - a word pronounced at a particular time, an apple located in the left corner of an image - the Fourier transform becomes a cumbersome tool."(Mallat, Wavelet Tour)

In astrophysics we are interested on finding the sources(Pierre Auger) and on eliminating them(WMAP).

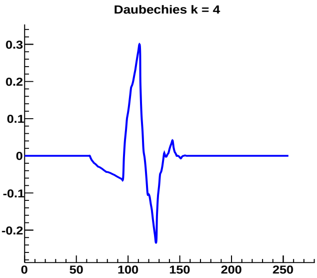


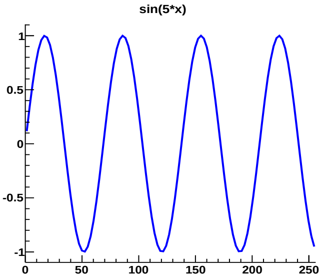
What are wavelets?



Wavelets in one dimension

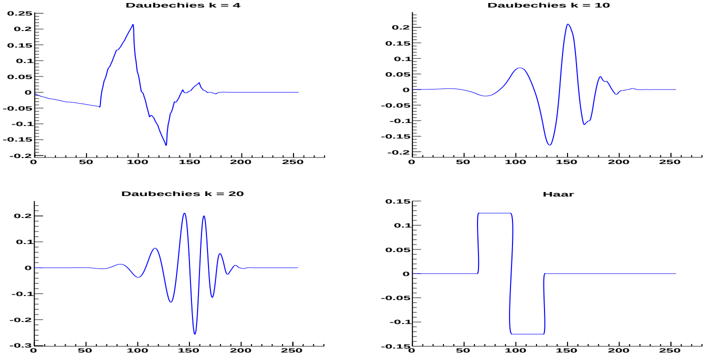
Functions with localization on frequency and time domain, whereas sines and cosines have localization only in frequency domain.





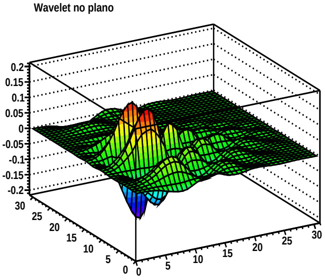
Various possibles wavelets to choose

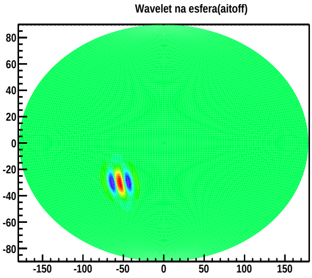
Some wavelets of the Daubechies family



Wavelets in two dimensions

- Defined on the plane(useful for images analysis).
- Defined on the sphere(data collected in all directions, WMAP, Pierre Auger).

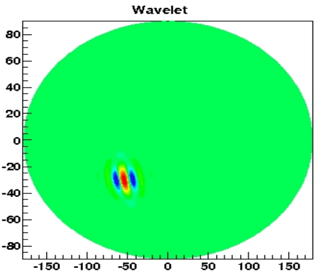


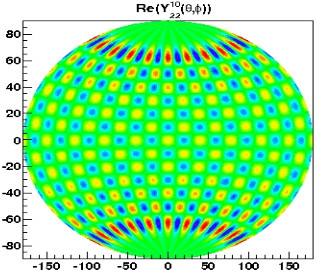


Why wavelet and not harmonic analysis?

Cosmic ray sources will manifest themselves as point-like source in event maps(limited by angular resolution) or maybe stretched due to the deflection of their trajectories by galactic and inter-galactic magnetic fields, so we need local functions to look for them.

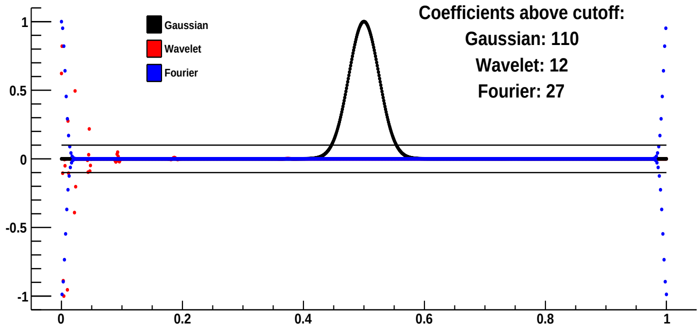
- Wavelets: Local.
- Spherical harmonics: Global





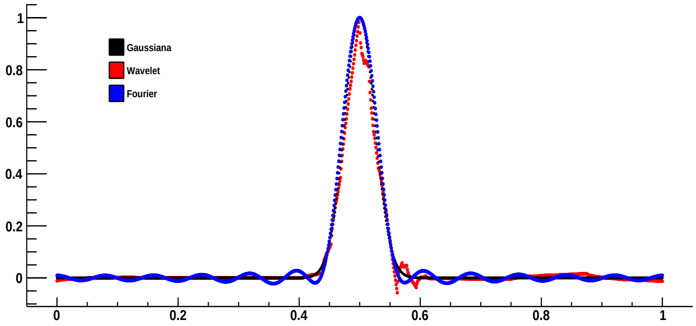
Fourier, wavelet or coordinate domain?

Representation of a Gaussian in Wavelet, Fourier and coordinate domain. We need less coefficients in wavelet domain to represent it!



Reconstruction

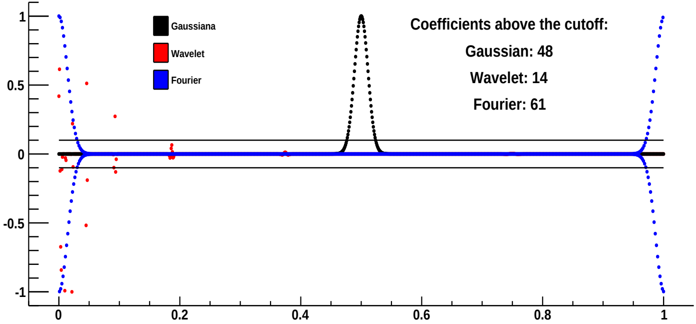
Wavelet reconstruction is more precise, since less coefficients are needed to represent it.



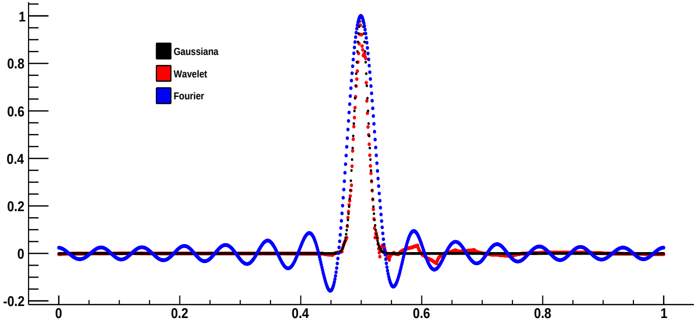
As the Gaussian gets narrower ...

Fourier, wavelet or coordinate domain?

We need four times more Fourier coefficients than wavelet coefficients.



Fourier reconstruction becomes really bad.



Complexity of the algorithm

- Coordinate domain $O(B^5)$.

$$C(g) = \int_{S^2} f(\omega)\overline{\Lambda(g)h(\omega)}d\omega$$

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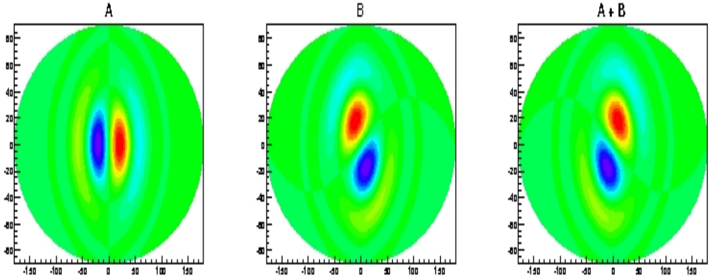
- Harmonic domain $O(B^4)$.

$$W(\alpha, \beta, \gamma) = \sum_{l=0}^{B-1} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \sum_{u=-l}^{l} \overline{a_{lm}} b_{ln} B^l_{mnu} e^{im\alpha} e^{iu\beta} e^{in\gamma}$$

$O(B^4) \rightarrow O(B^3)$: Steerable Wavelets

Using steerable wavelets the complexity is reduced to $O(B^3)$ with $(N << B)$. N ralated to the wavelet band limit.



Benchmarks available

B	Package	64	128	256	512	1024
μ(MB)	S2DW	1.3	5.3	21	84	340
	SWAT	2	20	120	420	1600
τ(min)	S2DW	0.019	0.092	0.73	7	72
	SWAT	0.011	0.083	0.68	5.56	47.5

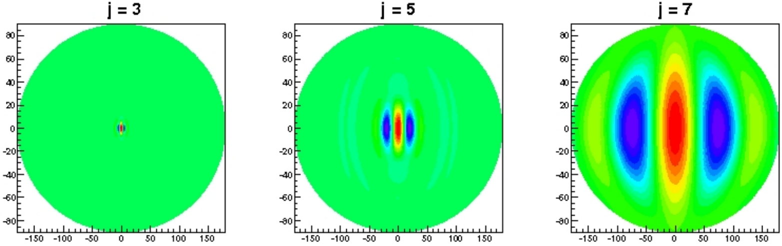
- SWAT measured on a 2.83 GHz Intel Core 2 Quad CPU.
- S2DW measured on a 2.20 GHz Intel Core 2 Duo CPU.

Dilations

Wavelet is split into a kernel and a directional part. Dilations do not affect directional properties.

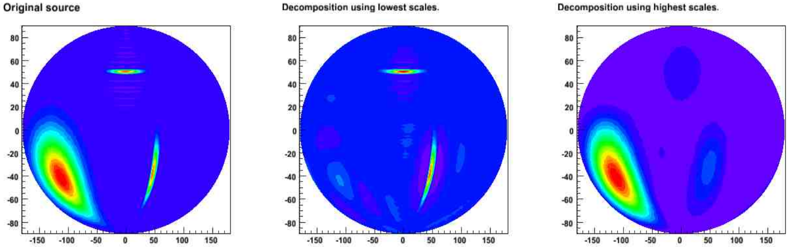
$$a_{lm} = K(l)S_{lm} \rightarrow K(2^j l)S_{lm}.$$

Example with J = 8:



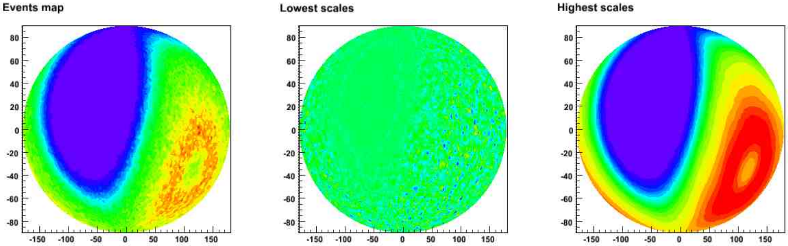
Removing structures using scale decomposition.

Without the knowledge of its location, we can remove structures decomposing the map in scales. The map in the middle is a reconstruction using the 5 lowest scales out of 7.



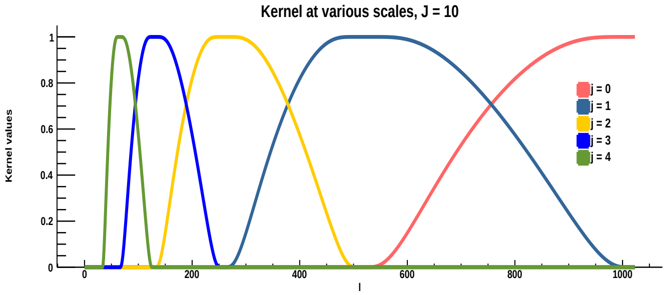
Removing structures using scale decomposition.

Example on Pierre Auger data: Managing exposure effects. The sum of the two graphs on the right is exactly the map on the left.

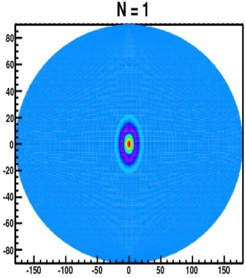


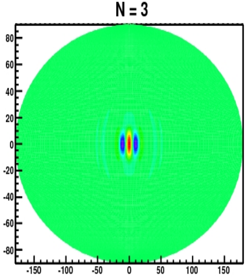
Kernel has compact support

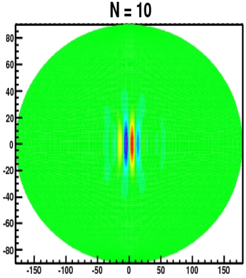
Only some frequencies are considered:



Controlling the azimuthal band limit we can control directional properties.

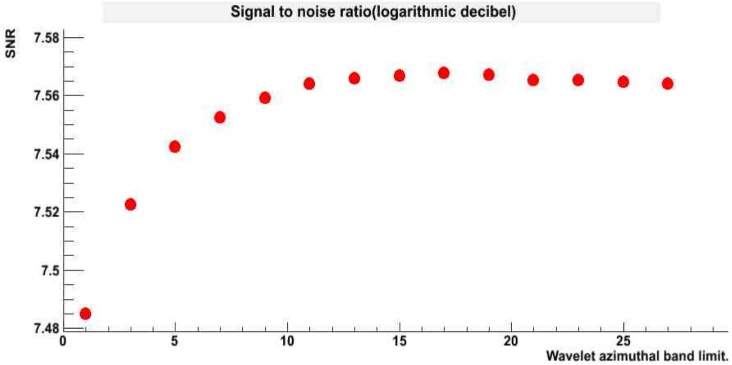






Application: increase SNR.

We can achieve higher signal to noise ratios if the wavelet correlates better with the source.



Denoising procedure

$f(\theta, \phi)$

1. From Pierre Auger data, we calculate a map(Healpix).

Denoising procedure

$f(\theta, \phi)$

$\downarrow$

a_{lm}

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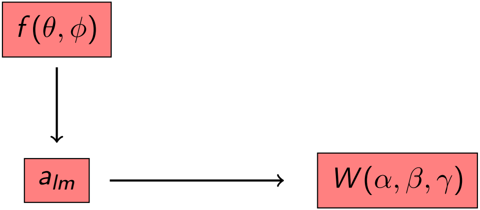
2. Healpix is used to calculate SH.



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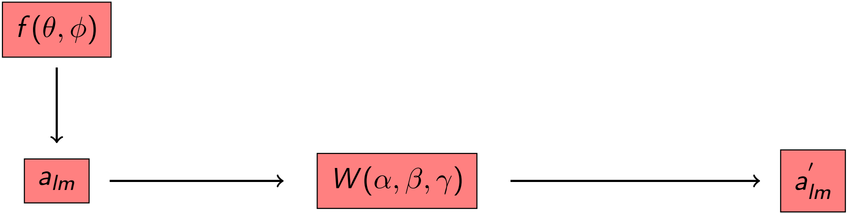

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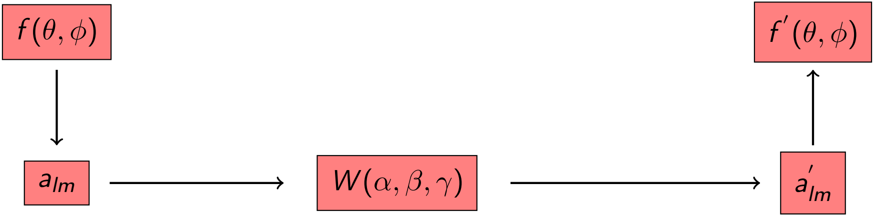

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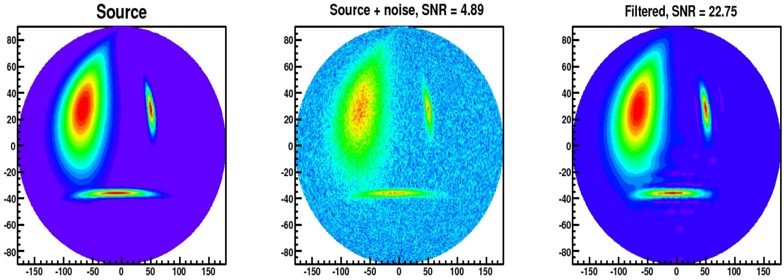
Denoising

Donoho and Johnstone threshold value:

$$T = \sigma\sqrt{2\log M}, \quad SNR = 20\log_{10}\left(\frac{\sigma_s}{\sigma_n}\right)$$

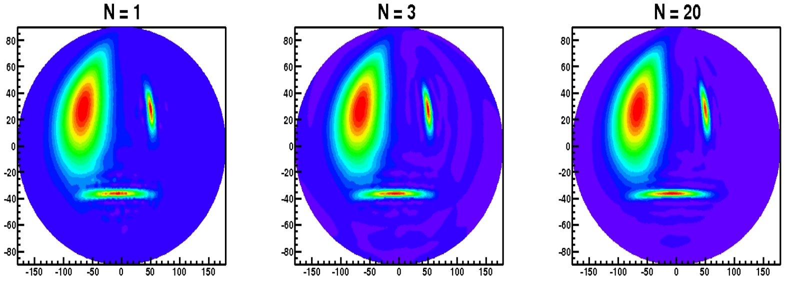
Where σ is the noise RMS and M the number of points. Denoising Gaussian beams immersed in a Gaussian noise:



Signal to noise ratio (SNR)

For different values of N we can achieve higher SNR depending on the structures on the map:

	$N = 1$	$N = 3$	$N = 10$	$N = 20$
SNR	21.99	23.91	25.13	25.23



Conclusion

Wavelets:

- Another tool on "spherical data" analysis.
- Enhance sparsity.
- Useful for denoising, calcule of anisotropy, etc.

Code:

- Integrated with ROOT.
- Objects can be saved to root files.
- Hard work done by FFTW.
- Easily loaded in Cint(good!).
- Healpix code included in the build system.

Available at: `http://www.ifi.unicamp.br/~mzimbres`